Tensor network exercises

Māris Ozols

September 9, 2019

1 Tensor network formalism

Snakes (tensor networks explained by Antoine de Saint-Exupéry)[1]

Let $|\Phi\rangle_{\mathsf{AB}} := \sum_{i=1}^{d} |i\rangle_{\mathsf{A}} \otimes |i\rangle_{\mathsf{B}}$.

Exercise 1. *Stretching a snake straight:*

$$\operatorname{Tr}_{\mathsf{B}} |\Phi\rangle\langle\Phi|_{\mathsf{AB}} = I_{\mathsf{A}}.$$

Exercise 2. *Snake equation:*

$$\begin{aligned} I &= \big(I_{\mathsf{A}} \otimes \langle\Phi|_{\mathsf{BC}}\big) \cdot \big(|\Phi\rangle_{\mathsf{AB}} \otimes I_{\mathsf{C}}\big) \\ &= \big(\langle\Phi|_{\mathsf{AB}} \otimes I_{\mathsf{C}}\big) \cdot \big(I_{\mathsf{A}} \otimes |\Phi\rangle_{\mathsf{BC}}\big). \end{aligned}$$

Exercise 3 (Transpose). *A snake which has eaten an elephant:*

$$\begin{aligned} E^{\mathsf{T}} &= (I_{\mathsf{A}} \otimes \langle\Phi|_{\mathsf{BC}}) \cdot (I_{\mathsf{A}} \otimes E_{\mathsf{B}} \otimes I_{\mathsf{C}}) \cdot (|\Phi\rangle_{\mathsf{AB}} \otimes I_{\mathsf{C}}) \\ &= (\langle\Phi|_{\mathsf{AB}} \otimes I_{\mathsf{C}}) \cdot (I_{\mathsf{A}} \otimes E_{\mathsf{B}} \otimes I_{\mathsf{C}}) \cdot (I_{\mathsf{A}} \otimes |\Phi\rangle_{\mathsf{BC}}). \end{aligned}$$

Exercise 4 (Sliding). *An elephant ending upside-down while being digested by a snake:*

$$(E_{\mathsf{A}} \otimes I_{\mathsf{B}})|\Phi\rangle_{\mathsf{AB}} = (I_{\mathsf{A}} \otimes E_{\mathsf{B}}^{\mathsf{T}})|\Phi\rangle_{\mathsf{AB}}.$$

Exercise 5 (Trace). *A snake which has eaten an elephant and then its own tail:*

$$\begin{aligned} \operatorname{Tr} E &= \langle\Phi|_{\mathsf{AB}} \cdot (E_{\mathsf{A}} \otimes I_{\mathsf{B}}) \cdot |\Phi\rangle_{\mathsf{AB}} \\ &= \langle\Phi|_{\mathsf{AB}} \cdot (I_{\mathsf{A}} \otimes E_{\mathsf{B}}) \cdot |\Phi\rangle_{\mathsf{AB}}. \end{aligned}$$

Exercise 6 (Cyclic property of trace). *If a snake eats two elephants and then its own tail, it can digest them in a cyclic fashion:*

$$\operatorname{Tr}(EF) = \operatorname{Tr}(FE).$$

[1] No animals should be harmed during these exercises.

Partial trace

Let $\mathrm{SWAP}_{\mathsf{AB}} := \sum_{i,j=1}^{d} (|i\rangle_{\mathsf{A}} \otimes |j\rangle_{\mathsf{B}}) \cdot (\langle j|_{\mathsf{A}} \otimes \langle i|_{\mathsf{B}})$.

Exercise 7. *A Siamese snake which has eaten a giraffe and then one of its own tails:*

$$(I_{\mathsf{A}} \otimes \langle\Phi|_{\mathsf{BC}}) \cdot (G_{\mathsf{AB}} \otimes I_{\mathsf{C}}) \cdot (I_{\mathsf{A}} \otimes |\Phi\rangle_{\mathsf{BC}}) = \mathrm{Tr}_{\mathsf{B}}\, G_{\mathsf{AB}}.$$

Exercise 8. *Partial trace of a rank-1 matrix via swap:*

$$\big(I_{\mathsf{A}} \otimes \langle\varphi|_{\mathsf{BC}}\big) \cdot \big(\mathrm{SWAP}_{\mathsf{AB}} \otimes I_{\mathsf{C}}\big) \cdot \big(I_{\mathsf{A}} \otimes |\psi\rangle_{\mathsf{BC}}\big) = \mathrm{Tr}_{\mathsf{B}}\, |\psi\rangle\langle\varphi|_{\mathsf{AB}}.$$

Exercise 9. *Partial trace identity of four rank-1 matrices:*

$$\mathrm{Tr}\Big[\Big(\mathrm{Tr}_{\mathsf{A}}\, |\alpha\rangle\langle\beta|_{\mathsf{AB}}\Big) \cdot \Big(\mathrm{Tr}_{\mathsf{A}}\, |\gamma\rangle\langle\delta|_{\mathsf{AB}}\Big)\Big] = \mathrm{Tr}\Big[\Big(\mathrm{Tr}_{\mathsf{B}}\, |\alpha\rangle\langle\delta|_{\mathsf{AB}}\Big) \cdot \Big(\mathrm{Tr}_{\mathsf{B}}\, |\gamma\rangle\langle\beta|_{\mathsf{AB}}\Big)\Big].$$

Partial transpose

For a composite system AB, the *partial transpose* on register A is defined as follows. If $M_{\mathsf{AB}} = A_{\mathsf{A}} \otimes B_{\mathsf{B}}$ then $M_{\mathsf{AB}}^{\Gamma} := A_{\mathsf{A}}^{\mathsf{T}} \otimes B_{\mathsf{B}}$, and we extend this to general M_{AB} by linearity.

Exercise 10. *Untwisting snakes:*

$$\mathrm{SWAP}^{\Gamma} = |\Phi\rangle\langle\Phi|.$$

Permutations

Exercise 11. *Cyclic shift from swaps: let $R := (\mathrm{SWAP}_{\mathsf{AB}} \otimes I_{\mathsf{C}}) \cdot (I_{\mathsf{A}} \otimes \mathrm{SWAP}_{\mathsf{BC}})$ and show that $R^3 = I_{\mathsf{ABC}}$.*

Let $Q_\pi \in \mathrm{U}(d^n)$ denote the unitary operation that permutes n qudits according to the permutation $\pi \in \mathrm{S}_n$. Let ρ be any qudit state.

Exercise 12. *Let $\tau \in \mathrm{S}_n$ denote the cyclic shift of n elements. Show that*

$$\mathrm{Tr}\big(Q_\tau \rho^{\otimes n}\big) = \mathrm{Tr}\,\rho^n \qquad \text{and} \qquad \mathrm{Tr}_{2,\ldots,n}\big(Q_\tau \rho^{\otimes n}\big) = \rho^n.$$

Exercise 13. *Let $\pi \in \mathrm{S}_n$ be an arbitrary permutation. Show that*

$$\mathrm{Tr}\big(Q_\pi \rho^{\otimes n}\big) = \prod_{c \in \pi} \mathrm{Tr}\,\rho^{|c|}$$

where c ranges over all cycles of π and $|c|$ denotes the length of the cycle c.

Exercise 14. *Let $U_{\mathsf{AB}} = \sqrt{\lambda}\, I_{\mathsf{AB}} + i\sqrt{1-\lambda}\, \mathrm{SWAP}_{\mathsf{AB}}$ for some $\lambda \in [0,1]$. Verify that U_{AB} is unitary and derive a simple expression for the state*

$$\mathrm{Tr}_{\mathsf{B}}\Big[U_{\mathsf{AB}}(\rho_{\mathsf{A}} \otimes \sigma_{\mathsf{B}})U_{\mathsf{AB}}^{\dagger}\Big]$$

just in terms of λ and the two input states ρ and σ.

2 Tensor networks and open quantum systems

Vectorization

Let $\operatorname{vec}(M) := (M_{\mathsf{A}} \otimes I_{\mathsf{B}})|\Phi\rangle_{\mathsf{AB}}$.

Exercise 1. *Vectorization identity:*

$$(A \otimes B) \cdot \operatorname{vec}(X) = \operatorname{vec}(AXB^{\mathsf{T}}).$$

Exercise 2. *More vectorization:*

$$(I \otimes B) \cdot \operatorname{vec}(A) = (A \otimes I) \cdot \operatorname{vec}(B^{\mathsf{T}}).$$

Exercise 3. *Vectorization and partial trace:*

$$\operatorname{Tr}_{\mathsf{B}}\Big(\operatorname{vec}(X)_{\mathsf{AB}} \operatorname{vec}(Y)^{\dagger}_{\mathsf{AB}}\Big) = XY^{\dagger}, \qquad \operatorname{Tr}_{\mathsf{A}}\Big(\operatorname{vec}(X)_{\mathsf{AB}} \operatorname{vec}(Y)^{\dagger}_{\mathsf{AB}}\Big) = X^{\mathsf{T}}\bar{Y}.$$

Exercise 4 (Purification)**.** *The canonical purification of a state ρ is*

$$|\psi\rangle_{\mathsf{AB}} := \big(\sqrt{\rho}_{\mathsf{A}} \otimes I_{\mathsf{B}}\big)|\Phi\rangle_{\mathsf{AB}} = \operatorname{vec}(\sqrt{\rho}).$$

Verify that $\langle\psi|\psi\rangle = 1$ and $\operatorname{Tr}_{\mathsf{B}} |\psi\rangle\langle\psi|_{\mathsf{AB}} = \rho_{\mathsf{A}}$.

Recall that any $d_{\mathsf{A}} \times d_{\mathsf{B}}$ complex matrix M has a *singular value decomposition*

$$M = UDV$$

where $U \in \mathrm{U}(d_{\mathsf{A}})$, $V \in \mathrm{U}(d_{\mathsf{B}})$, and D is a $d_{\mathsf{A}} \times d_{\mathsf{B}}$ matrix with non-negative diagonal entries and zero off-diagonal entries.

Exercise 5 (Schmidt decomposition)**.** *Show that any bipartite pure state $|\psi\rangle_{\mathsf{AB}}$ admits a decomposition of the following form:*

$$|\psi\rangle_{\mathsf{AB}} := \sum_{i=1}^{r} s_i \, |u_i\rangle_{\mathsf{A}} \otimes |v_i\rangle_{\mathsf{B}}$$

where $r := \min\{d_{\mathsf{A}}, d_{\mathsf{B}}\}$, and $\{|u_1\rangle, \dots, |u_d\rangle\}$ and $\{|v_1\rangle, \dots, |v_d\rangle\}$ are orthonormal sets of vectors in $\mathbb{C}^{d_{\mathsf{A}}}$ and $\mathbb{C}^{d_{\mathsf{B}}}$, respectively.

Quantum channels

Exercise 6. *Find a superoperator that is Hermitian-preserving but not positivity-preserving.*

Exercise 7. *A canonical example of a superoperator that is positive but not completely positive is the transpose $\mathcal{T}(A) := A^{\mathsf{T}}$. To prove that $\mathcal{E}$ is not completely positive, show that*

$$(\mathcal{T}_{\mathsf{A}} \otimes \mathcal{I}_{\mathsf{B}})(|\Phi\rangle\langle\Phi|_{\mathsf{AB}}) = \mathrm{SWAP}_{\mathsf{AB}}$$

where $\mathcal{I}_{\mathsf{B}}$ is the identity superoperator. To show that $\mathrm{SWAP}_{\mathsf{AB}}$ is not positive semidefinite, find a non-zero vector $|\psi\rangle_{\mathsf{AB}}$ such that

$$\mathrm{SWAP}_{\mathsf{AB}} \, |\psi\rangle_{\mathsf{AB}} = -|\psi\rangle_{\mathsf{AB}}.$$

Representations of quantum channels

- The *natural representation* of a quantum channel $\mathcal{E}$ from d_{in} to d_{out} dimensions is a linear operator
$$S_{\mathcal{E}} : \mathbb{C}^{d_{\text{in}}} \otimes \mathbb{C}^{d_{\text{in}}} \to \mathbb{C}^{d_{\text{out}}} \otimes \mathbb{C}^{d_{\text{out}}}$$
defined as follows: for any $d_{\text{in}} \times d_{\text{in}}$ complex matrix A,
$$S_{\mathcal{E}} \cdot \operatorname{vec}(A) = \operatorname{vec}(\mathcal{E}(A)).$$

- The *Choi-Jamiołkowski representation* of $\mathcal{E}$ is a linear operator
$$J_{\mathcal{E}} : \mathbb{C}^{d_{\text{out}}} \otimes \mathbb{C}^{d_{\text{in}}} \to \mathbb{C}^{d_{\text{out}}} \otimes \mathbb{C}^{d_{\text{in}}}$$
that can be defined in terms of $S_{\mathcal{E}}$ as follows:
$$\langle i, j | J_{\mathcal{E}} | k, l \rangle = \langle i, k | S_{\mathcal{E}} | j, l \rangle,$$
for all $i, k \in \{1, \dots, d_{\text{out}}\}$ and $j, l \in \{1, \dots, d_{\text{in}}\}$.

- A *Kraus representation* of $\mathcal{E}$ is given by a collection of $r = \operatorname{rank}(J_{\mathcal{E}})$ operators
$$K_1, \dots, K_r : \mathbb{C}^{d_{\text{in}}} \to \mathbb{C}^{d_{\text{out}}}$$
obtained from the spectral decomposition of $J_{\mathcal{E}}$:
$$J_{\mathcal{E}} = \sum_{i=1}^{r} \operatorname{vec}(K_i) \operatorname{vec}(K_i)^\dagger.$$

- A *Stinespring representation* of $\mathcal{E}$ is given by an isometry
$$A_{\mathcal{E}} : \mathbb{C}^{d_{\text{in}}} \to \mathbb{C}^{d_{\text{out}}} \otimes \mathbb{C}^{r}$$
that can be obtained from a Kraus representation of $\mathcal{E}$ as follows:
$$A_{\mathcal{E}} := \sum_{i=1}^{r} K_i \otimes |i\rangle.$$

Exercise 8. *For any quantum channel $\mathcal{E}$, show that*
$$(\mathcal{E}_{\mathsf{A}} \otimes \mathcal{I}_{\mathsf{B}})(|\Phi\rangle\langle\Phi|_{\mathsf{AB}}) = (J_{\mathcal{E}})_{\mathsf{AB}}.$$
Hence, if $\mathcal{E}$ is CP then $J_{\mathcal{E}} \geq 0$.

Exercise 9. *Verify that the Stinespring representation $A_{\mathcal{E}}$ of a quantum channel $\mathcal{E}$ is an isometry: $A_{\mathcal{E}}^\dagger A_{\mathcal{E}} = I$. Show that any quantum channel $\mathcal{E}$ can be expressed as*
$$\mathcal{E}(\rho) = \operatorname{Tr}_{\mathsf{B}}\left[U_{\mathsf{AB}}(\rho_{\mathsf{A}} \otimes |1\rangle\langle 1|_{\mathsf{B}}) U_{\mathsf{AB}}^\dagger\right],$$
for some unitary matrix U_{AB}, where $|1\rangle_{\mathsf{B}}$ is the first standard basis state of the r-dimensional register B where $r = \operatorname{rank}(J_{\mathcal{E}})$.

Exercise 10. *Show that the natural and Kraus representation of a quantum channel $\mathcal{E}$ are related as follows:*

$$S_{\mathcal{E}} = \sum_{i=1}^{r} K_i \otimes \bar{K}_i.$$

Exercise 11. *Show that $\mathcal{E}$ is positivity-preserving if and only if*

$$\mathrm{Tr}\Big[(J_{\mathcal{E}})_{\mathsf{AB}} S_{\mathsf{AB}}\Big] \geq 0$$

for any separable operator S_{AB}. Recall that S_{AB} is separable if and only if

$$S_{\mathsf{AB}} = \sum_i A^i_{\mathsf{A}} \otimes B^i_{\mathsf{B}}$$

for some $A^i, B^i \geq 0$.

3 Quantum circuits as tensor networks

Pauli matrices and Bell states

Pauli matrices are given by

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ = -iZX.$$

Bell states are given by

$$|\beta_{00}\rangle := \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \sqrt{2}|\Phi\rangle, \qquad |\beta_{01}\rangle := \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$
$$|\beta_{10}\rangle := \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \qquad |\beta_{11}\rangle := \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

For any $z, x \in \{0, 1\}$, the two are related as follows:

$$|\beta_{zx}\rangle = \frac{1}{\sqrt{2}} \operatorname{vec}(Z^z X^x).$$

Exercise 1 (Local conversion of Bell states). *Show that, for any $z, x \in \{0, 1\}$,*

$$|\beta_{zx}\rangle = (Z^z X^x \otimes I)\, |\beta_{00}\rangle = (I \otimes X^x Z^z)\, |\beta_{00}\rangle.$$

Exercise 2 (Pauli matrices and SWAP). *Verify that*

$$\frac{1}{2} \sum_{z,x \in \{0,1\}} Z^z X^x \otimes X^x Z^z = \frac{1}{2}(I \otimes I + X \otimes X + Y \otimes Y + Z \otimes Z) = \mathrm{SWAP},$$
$$\frac{1}{2} \sum_{z,x \in \{0,1\}} Z^z X^x \otimes Z^z X^x = |\Phi\rangle\langle\Phi|.$$

Quantum teleportation

Exercise 3 (Teleportation from SWAP). *Prove that*

$$|\psi\rangle_{\mathsf{A}} \otimes |\beta_{00}\rangle_{\mathsf{A'B}} = \frac{1}{2} \sum_{z,x \in \{0,1\}} |\beta_{zx}\rangle_{\mathsf{AA'}} \otimes X^x Z^z |\psi\rangle_{\mathsf{B}}.$$

Derive the teleportation protocol from this identity.

Exercise 4 (Teleportation from snakes). *Note that, for any $|\psi\rangle \in \mathbb{C}^2$,*

$$\big(\langle\beta_{00}|_{\mathsf{AA'}} \otimes I_{\mathsf{B}}\big) \cdot \big(|\psi\rangle_{\mathsf{A}} \otimes |\beta_{00}\rangle_{\mathsf{A'B}}\big) = \frac{1}{2}|\psi\rangle_{\mathsf{B}}.$$

Generalize this by finding a matrix P_{zx} such that, for all $z, x \in \{0, 1\}$,

$$\big(\langle\beta_{zx}|_{\mathsf{AA'}} \otimes I_{\mathsf{B}}\big) \cdot \big(|\psi\rangle_{\mathsf{A}} \otimes \operatorname{vec}(P_{zx})_{\mathsf{A'B}}\big) = \frac{1}{2}|\psi\rangle_{\mathsf{B}}.$$

Find a matrix Q_{zx} such that, for all $z, x \in \{0, 1\}$,

$$(Q_{zx})_{\mathsf{B}} \cdot \big(\langle\beta_{zx}|_{\mathsf{AA'}} \otimes I_{\mathsf{B}}\big) \cdot \big(|\psi\rangle_{\mathsf{A}} \otimes |\beta_{00}\rangle_{\mathsf{A'B}}\big) = \frac{1}{2}|\psi\rangle_{\mathsf{B}}.$$

Derive the teleportaiton protocol from this.

Exercise 5 (Entanglement swapping)**.** *Verify the following identity:*

$$|\beta_{00}\rangle_{\mathsf{ZA}} \otimes |\beta_{00}\rangle_{\mathsf{A'B}} = \frac{1}{2} \sum_{z,x \in \{0,1\}} |\beta_{zx}\rangle_{\mathsf{ZB}} \otimes |\beta_{zx}\rangle_{\mathsf{AA'}}.$$

(note the slightly different order of qubits on both sides). What happens when Alice ($\mathsf{AA'}$) and Bob (B) execute the teleportation protocol on state $|\beta_{00}\rangle_{\mathsf{ZA}}$ where Z is held by Zoey?

"Quantum software" and gate teleportation

Exercise 6 (Quantum software)**.** *Alice has invented a new quantum algorithm and implemented it on her quantum computer. Her algorithm is described by some $d \times d$ unitary U. Alice decides to sell it to make some money, so she opens a quantum software factory that produces the maximally entangled state $\frac{1}{\sqrt{d}}|\Phi\rangle_{\mathsf{AA'}}$ with U applied on the first register. This state effectively encode the unitary U_{A} since*

$$(U_{\mathsf{A}} \otimes I_{\mathsf{A'}}) \cdot \frac{1}{\sqrt{d}}|\Phi\rangle_{\mathsf{AA'}} = \frac{1}{\sqrt{d}} \operatorname{vec}(U)_{\mathsf{AA'}}.$$

Bob has a d-dimensional state $|\psi\rangle_{\mathsf{B}}$ on which he wants to run Alice's algorithm. He purchases one copy of Alice's state and measures the registers $\mathsf{A'B}$ in some basis that contains the maximally entangled state $\frac{1}{\sqrt{d}}|\Phi\rangle_{\mathsf{A'B}}$.

- *What is the probability that Bob gets the outcome corresponding to $\frac{1}{\sqrt{d}}|\Phi\rangle_{\mathsf{A'B}}$?*
- *What is his post-measurement state in this case?*
- *If U belongs to the Clifford group, how can Bob increase his success probability to 100%?*

Exercise 7 (Choi state as quantum software)**.** *To save memory, Alice optimizes her algorithm so that it discards unnecessary information. Her new algorithm is described by a quantum channel $\mathcal{E}$ from d_{in} to d_{out} dimensions. She comes up with a way of preparing a quantum state*

$$\rho_{\mathcal{E}} := c \cdot J_{\mathcal{E}}$$

for some constant $c \in \mathbb{R}$, where $J_{\mathcal{E}}$ is the Choi matrix of $\mathcal{E}$.

1. *What is the correct normalization constant c?*
2. *How can Bob execute Alice's algorithm $\mathcal{E}$ on his input state σ?*
3. *What is his probability of success?*

Simulation of quantum circuits

Exercise 8. *Consider the following tensor network: a* $2 \times n$ *rectangular grid with each vertex containing a tensor and each edge having dimension* 2*. What is the space and time complexity of contracting this tensor network if we*

1. *first contract the vertical, then the horizontal edges,*

2. *or the other way round?*