Quantum Circuit Decompositions: Additional material

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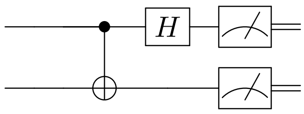
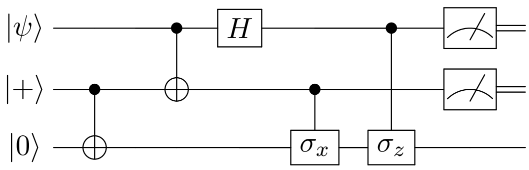
Exercises

Exercise 1. a) Consider the measurement outcomes when the Bell states are sent through the circuit



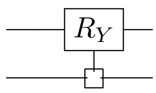
What does this measurement correspond to?

b) What does the following quantum circuit do?



[Hint: This circuit is closely related to a common quantum task.]

Exercise 2. Express the following gate as a matrix:



Exercise 3. If the Cosine-Sine decomposition method is used to decompose a 2-qubit unitary, how many CNOTs are in the final circuit?

Exercise 4. How many real parameters are needed to describe an isometry from m to n qubits?

Hence explain why an m to n isometry requires at least $\frac{1}{4}(2^{n+m+1} - 2^{2m} - 2n - m - 1)$ CNOTs.

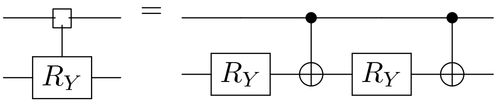
Exercise 5. In this exercise, we wish to show that for any single qubit unitary, U, there exists θ, ϕ and τ such that (up to global phase) $U = R_Z(\theta)R_Y(\phi)R_Z(\tau)$.

Show that an arbitrary single qubit unitary can be expressed in matrix form as $\begin{pmatrix} e^{i\frac{\alpha_1}{2}}\cos\frac{\beta}{2} & e^{i\frac{\alpha_3}{2}}\sin\frac{\gamma}{2} \\ e^{i\frac{\alpha_2}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha_4}{2}}\cos\frac{\gamma}{2} \end{pmatrix}$, with $0 \leq \beta \leq \pi$, $0 \leq \gamma \leq \pi$ and $\alpha_i \in \mathbb{R}$.

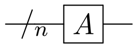
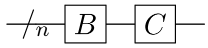
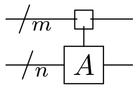
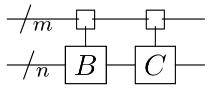
Explain why we only need implement the unitary up to global phase, and that this allows us to choose $\alpha_4 = -\alpha_1$. Having made this choice for α_4, show that orthogonality of the columns implies $\gamma = \beta$.

Hence establish the claim.

Exercise 6. Prove the following identity:

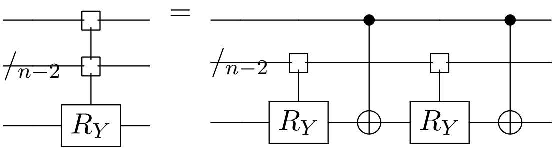


(∗)

The quantum multiplexor property states that if  =  is a circuit identity (where A, B and C have some free parameters) then so is  =  .

Prove this for the case $m = 1$.

Show that the application of the quantum multiplexor property to (∗) gives



Exercise 7. Show how to implement an arbitrary 2-outcome POVM via an isometry.

Additional Proofs

Lemma 1. *For all unitaries U_0 and U_1 (of fixed dimension), there exist unitaries U and V, and real numbers $\{\phi_i\}_i$ such that*

$$|0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1 = (\mathbb{1} \otimes U) \cdot (\sum_i R_Z(\phi_i) \otimes |i\rangle\langle i|) \cdot (\mathbb{1} \otimes V)\,.$$

Proof. We can express this in another way:

$$U_0 \oplus U_1 = (U \oplus U) \cdot (D \oplus D^\dagger) \cdot (V \oplus V)\,,$$

where D is a diagonal unitary, or equivalently, $U_0 = UDV$ and $U_1 = UD^\dagger V$.

Note that the unitary $U_0 U_1^\dagger$ is normal[1], so can be diagonalized by a unitary matrix, i.e. there exists unitary U and diagonal matrix C such that $U_0 U_1^\dagger = UCU^\dagger$. Note that C must be unitary, since $C = U^\dagger U_0 U_1^\dagger U$, which is the product of unitaries.

Taking $V = C^{\frac{1}{2}} U^\dagger U_1$, it follows that $UC^{\frac{1}{2}} V = UCU^\dagger U_1 = U_0$ and $U(C^\dagger)^{\frac{1}{2}} V = U(C^\dagger)^{\frac{1}{2}} C^{\frac{1}{2}} U^\dagger U_1 = U_1$.[2] Thus, choosing $D = C^{\frac{1}{2}}$ we obtain the claimed relation. □

[1] Recall that A is normal means $AA^\dagger = A^\dagger A$

[2] Note that $(C^\dagger)^{\frac{1}{2}} C^{\frac{1}{2}} = \mathbb{1}$.

Lemma 2 (Cosine-Sine Decomposition (CSD))**.** *Let U be a $d \times d$ unitary matrix for d even. There exist $\frac{d}{2} \times \frac{d}{2}$ unitary matrices A_1, A_2, B_1 and B_2 and real diagonal matrices C and S satisfying $C^2 + S^2 = \mathbb{1}_{d/2}$ such that*

$$U = \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix},$$

where 0 represents the $\frac{d}{2} \times \frac{d}{2}$ zero matrix.

The proof relies on two well-known matrix decompositions: the singular value decomposition (SVD), and the QR-decomposition.

Lemma 3 (SVD)**.** *For any square matrix M, there exist unitaries V and W, and a real diagonal matrix D such that $M = VDW$.*

Lemma 4 (QR-decomposition)**.** *For any square matrix M, there exist a unitary Q and a right triangular matrix R such that $M = QR$. Furthermore, R can be chosen such that its diagonal entries are all non-negative.*

Note that R is a right triangular matrix iff $R_{i,j} = 0$ when $i > j$, e.g., $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$. Similarly, every matrix M has an LQ-decomposition, in which $M = LQ$ with L lower triangular with non-positive diagonal entries, and Q unitary.

Proof of Lemma 2. Let us write

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

noting that $U_1, \ldots, U_4$ are not in general unitary. Consider then

$$M := \begin{pmatrix} A_1^\dagger & 0 \\ 0 & B_1^\dagger \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} A_2^\dagger & 0 \\ 0 & B_2^\dagger \end{pmatrix} = \begin{pmatrix} A_1^\dagger U_1 A_2^\dagger & A_1^\dagger U_2 B_2^\dagger \\ B_1^\dagger U_3 A_2^\dagger & B_1^\dagger U_4 B_2^\dagger \end{pmatrix}.$$

We choose A_1 and A_2 to be such that $A_1 D A_2 = U_1$ (via the SVD), i.e., $A_1^\dagger U_1 A_2^\dagger =: C$ is diagonal. We then choose B_1 to be Q in the QR-decomposition of $U_3 A_2^\dagger$, so that $B_1^\dagger U_3 A_2^\dagger =: R$ is upper triangular with non-negative entries on its diagonal (via the QR-decomposition). Likewise, we choose B_2 to be Q in the LQ-decomposition of $A_1^\dagger U_2$ so that $A_1^\dagger U_2 B_2^\dagger =: L$ is lower triangular with non-positive entries on its diagonal. For simplicity, we will assume that all the diagonal elements of R and L are non-zero. This isn't necessary, but the more general case complicates the proof without adding insight.

We have

$$M = \begin{pmatrix} C & L \\ R & M_4 \end{pmatrix},$$

where we use $M_4 = B_1^\dagger U_4 B_2^\dagger$ for brevity. Since it is the product of three unitary matrices, M is unitary and hence its columns must be orthogonal. It follows that R is diagonal. (For example, if we write the first two columns of M as $(c_{11}, 0, \ldots, 0, r_{11}, 0, \ldots, 0)$ and $(0, c_{22}, 0, \ldots, 0, r_{12}, r_{22}, 0, \ldots, 0)$, then their inner product is $r_{11}r_{12}$. Therefore (due to our earlier assumption on the diagonal entries of R) we must have $r_{12} = 0$.) Likewise, considering the orthogonality of the rows, L is diagonal. In order that the columns are normalised, we require $C^2 + R^2 = \mathbb{1}$, and similarly for the rows, $C^2 + L^2 = \mathbb{1}$, which can be achieved if we take $L = -R$. Orthogonality of the columns means that $M_4 = C$, and hence if we define $S := R$, we have $M = \begin{pmatrix} C & -S \\ S & C \end{pmatrix}$, which establishes the CSD. □