Work extraction and information

September 12, 2019

Rules

We present the contesting teams of 2-4 players with one task about work extraction in quantum systems. It consists of few steps of increasing difficulty; the teams will be judged upon the number of subtasks successfully solved, as well as the efficiency of solutions. Each solution has to be accompanied by a written explanation of methods used and implemented, and the compilation guide.

We do not restrict you to any one programming language. You are free to use whichever you prefer (Python, ProjectQ, Q#, Qiskit, assembly, C++, ... just no FORTRAN please). At the end of the day, upload all files to a private github repo and share it with the account @squidschools.

We hope you enjoy the task!:)

The task at hand will mainly encompass a research area called *quantum thermodynamics*. It studies the connections between quantum physics and thermodynamics; reviews of the field can be found in [1], [2].

Quick overview: work and information

Landauer's principle. In 1961, Rolf Landauer has formulated a physical principle that outlines a fundamental limit to the heat released when one bit of information is erased, i.e. reset from an unknown state back to 0, like when you format a hard drive [3]. It sets the lower bound to this heat as $kT \ln 2$ for the erasure process carried out at temperature T. Originally formulated for classical bits, the principle can be also generalized for the quantum case [4]. Thus, the erasure of information has a work cost - and our goal is to extract it (1).

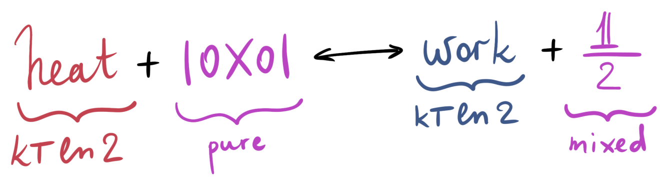

Figure 1: **An intuition behind Landauer's principle.** On the left hand side there is a qubit in a pure state, which corresponds to one bit of information, and heat from the heat bath; on the right hand side, the qubit is in a fully mixed state, but heat transformed into work which we can use.

Szilard's engine. One of the classical examples of such work extraction was proposed by Leo Szilard [5]. He showed that given having prior information about a system allows for work extraction (Figure 2). In his thought experiment he has a particle in a box in contact with a heat bath at temperature T with a shutter between the two halves of the box. If one knows where the particle is, for instance, in the right compartment, one can let it expand to the left and use the moving shutter to extract work, for example, drag a weight. By letting the piston move very slowly, such that the particle can thermalize with the environment outside the box at temperature T, and integrating the work gained over the volume expansion of $V/2$, we obtain:

$$W = \int_{V/2}^{V} p dV = \int_{V/2}^{V} \frac{kT}{V} dV = kT \ln 2.$$

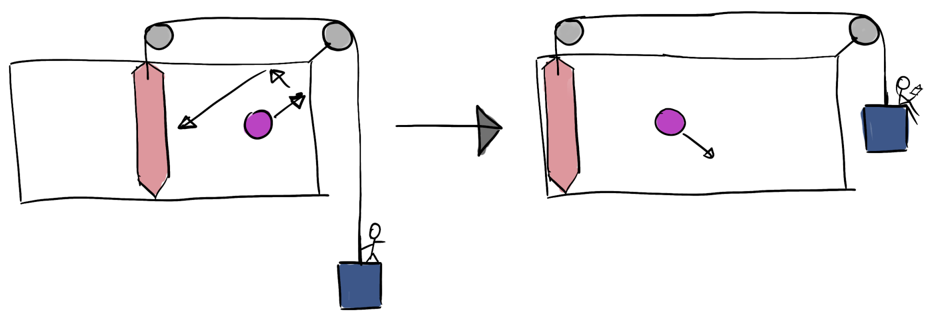

Figure 2: **Szilard's engine.** If we know the location of the particle prior to the experiment (here it is in the right compartment of the box), we can extract work by letting it expand to the left and using the moving shutter as pictured above.

A semi-classical example can be seen on Figure 3.

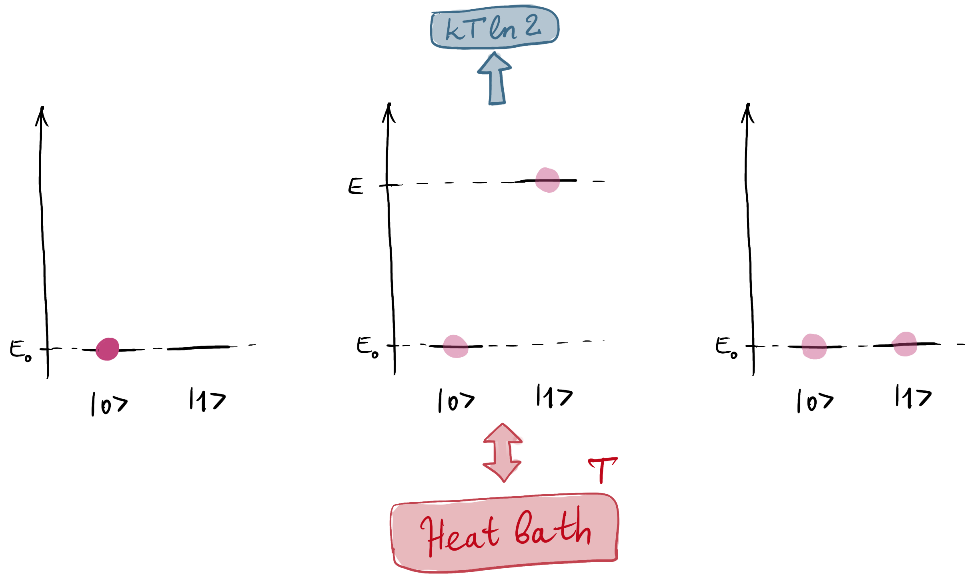

Figure 3: **Work extraction by energy levels manipulation.** To just give an idea of how this applies to the quantum case, an example of how the work can be extracted from a quantum system where one can raise and lower energy levels, can be constructed [6] (Figure 3). Let us suppose that we have a qubit in a pure state $|0\rangle$ with energy E_0. The energy of the excited state $E \gg E_0$. After connecting the qubit to a heat bath at temperature T there is small probability $p(E)$ that the qubit can be found in the excited state $\langle 1|$, where the probability is given by Gibbs distribution

$$p(E) = \frac{1}{1 + e^{(E-E_0)/kT}}.$$

Lowering the energy of the excited state in infinitesimal steps results in a work gain at a cost of Landauer's limit:

$$\lim_{E\to\infty} \int_{E_0}^{E} p(E')dE' = kT \ln 2$$

This analysis is still semi-classical, as here we do not explicitly model a device where we store the extracted work.

In the next few sections, we give an intuition of a different protocol which does not involve moving energy levels of the system, and can be described as a unitary transformation. It also allows for storing the work in a separate system, which ensures it is accessible for use. First, we describe the main elements of the experimental setting, and then give an idea of the key transformation enabling work gain.

The setting

Here we describe three founding parts of the experimental setting: the information battery, the thermal qubit and the energy battery (storage).

The information battery. The information battery B is a device where we store information (in the sense of a nearly-pure quantum state), but not energy (because the Hamiltonian is degenerate: $H_B = 0$). In the beginning of our task, it consists of one degenerate ϵ-pure qubit:

$$\rho_B^{(0)} = \big(\epsilon|R\rangle\langle R| + (1-\epsilon)|L\rangle\langle L|\big),\ \epsilon \ll 1,$$

where $\{|L\rangle, |R\rangle\}$ is the standard basis of the qubit.

The thermal qubits. The extraction of work is only possible via coupling of the information battery to a thermal bath at a constant temperature T. We portray this interaction as an interaction with a non-degenerate thermal qubit Q_i with an energy gap Δ_i, which can be described by a Hamiltonian $H_{Q_i} = \Delta_i|1\rangle\langle 1|$ ($|0\rangle$ goes for the ground state of the thermal qubit, and $|1\rangle$ for the excited). After the qubit *thermalizes* - reaches a thermal equilibrium with the heat bath — its state can be written as:

$$\rho_{Q_i}^{(0)} = p_i|1\rangle\langle 1| + (1-p_i)|0\rangle\langle 0|,\ \text{where}\ \ p_i = \frac{1}{1+e^{\Delta_i/kT}},$$

where $p(E)$ is given by Gibbs distribution.

The energy battery. The energy battery is where we store the acquired energy. It can be modelled as a quantum system S consisting of infinitely many equidistant energy levels with a Hamiltonian $H_S = \sum_{-N}^{+N} n\cdot\delta\cdot|n\rangle\langle n|,\ N \gg 1$. We will further take the initial state of the storage system as a pure state: $\rho_S^{(0)} = |0\rangle\langle 0|$.

Just before we go on further to qualitatively outline the protocol for extracting work from the information battery modelled as an almost pure state, let us elaborate on which operations we consider allowed in this setting. First, we let an operation to be carried out on composite system consisting of an information battery B, a thermal qubit Q_i and an energy battery (storage) S. These operations have to be energy-conserving and reversible, i.e. they need to be unitary and commute with the global Hamiltonian $[U, H_{global}] = 0$, where H_{global} is:

$$H_{global} = H_B \otimes \mathbb{I}_{Q_i} \otimes \mathbb{I}_S + \mathbb{I}_B \otimes H_{Q_i} \otimes \mathbb{I}_S + \mathbb{I}_B \otimes \mathbb{I}_{Q_i} \otimes H_S.$$

Secondly, we allow for the qubits to thermalize, namely, undergo a process described above. This set of allowed operations is typical for resource theories of quantum thermodynamics [1].

The work extraction protocol for a qubit in an almost pure state

Now let us proceed with the protocol, which is based on the idea proposed in [7]; the process is pictured on the Figure 4. It is a unitary, which can be formally written as follows (Δ_i is the energy gap of Q_i, and δ is the energy gap of S):

$$U_i = \sum_n \left(|L\rangle_B|1\rangle_{Q_i}|n\rangle_S\langle R|_B\langle 0|_{Q_i}\langle n+\frac{\Delta_i}{\delta}|_S + |R\rangle_B|0\rangle_{Q_i}|n+\frac{\Delta_i}{\delta}\rangle_S\langle L|_B\langle 1|_{Q_i}\langle n|_S\right)$$

$$+(|R\rangle_B|1\rangle_{Q_i}\langle R|_B\langle 1|_{Q_i} + |L\rangle_B|0\rangle_{Q_i}\langle L|_B\langle 0|_{Q_i})\otimes \mathbb{I}_S$$

Qualitatively, the action of the unitary can be seen as a list of conditions below:

1. If the information battery and the thermal qubit are in the state $|L\rangle_B|1\rangle_{Q_i}$, we take them to the state $|R\rangle_B|0\rangle_{Q_i}$, and lift the energy battery state from $|n\rangle_S$ to $|n+\frac{\Delta_i}{\delta}\rangle_S$, and vice verse;
2. If the information battery and the thermal qubit are in the state$|L\rangle_B|0\rangle_{Q_i}$ or $|R\rangle_B|1\rangle_{Q_i}$, then nothing happens.

If we apply this unitary to the initial state of the composite system $\rho^{(0)} = \rho_B^{(0)}\otimes\rho_{Q_i}^{(0)}\otimes\rho_S^{(0)}$, we arrive to the state $\rho^{(1)} = U\rho^{(0)}U^\dagger$. To make an interesting observation, let us look at the final states of the reduced subsystems B, Q and S:

$$\rho_B^{(1)} = tr_{Q_iS}(\rho^{(1)}) = p|R\rangle\langle R| + (1-p)|L\rangle\langle L|;$$

$$\rho_{Q_i}^{(1)} = tr_{BS}(\rho^{(1)}) = \left(\epsilon|1\rangle\langle 1| + (1-\epsilon)|0\rangle\langle 0|\right);$$

$$\rho_S^{(1)} = tr_{BQ_i}(\rho^{(1)}) = (1-p)\epsilon|-\frac{\Delta_i}{\delta}\rangle\langle-\frac{\Delta_i}{\delta}| + [p\epsilon + (1-p)(1-\epsilon)]|0\rangle\langle 0| + p(1-\epsilon)|\frac{\Delta_i}{\delta}\rangle\langle\frac{\Delta_i}{\delta}|.$$

First, our transformation simply swapped the states of the thermal qubit and the information battery; secondly, there is a non-zero probability for the energy battery to actually lower its energy and end up in the state $|-1\rangle\langle -1|$. Intuitively, in order for the observed energy of the battery to have an increase, the probability to find it in the state $|1\rangle\langle 1|$ has to be no less than the probability to find it in the state with reduced energy $|-1\rangle\langle -1|$. A simple calculation then yields $p \geq \epsilon$; this will become useful later, when we turn to the case of N thermal qubits.

The case of N thermal qubits. The protocol above can be extended for the case of N thermal qubits $Q_1, \dots, Q_N$ [8]. Here we give a brief idea of how the protocol proceeds and what is the intuition behind it.

For simplicity let us suppose that the qubits are ordered by how wide their energy gap is, that is, if we denote the energy gap of the kth qubit as Δ_k, then $\delta_1 > \delta_2 > \cdots > \delta_N$ ($p_1 < p_2 < \cdots < p_N$). Then the work extraction process can be described by a sequence of unitary transformations $\{U_k\}_{k=\overline{1,n}}$, where each transformation U_k exchanges the following states:

$$U_k : |L\rangle_B|1\rangle_{Q_k}|n\rangle_S \leftrightarrow |R\rangle_B|0\rangle_{Q_k}|n+\frac{\Delta_k}{\delta}\rangle_S,$$

where δ is the energy gap of the energy battery (Figure 5). Running the protocol for a sequence where each next qubit has a narrower gap (and thus larger p), allows us to keep the information battery "purer" than the thermal qubit for each stage of the protocol. This ensures the net increase of energy stored in the energy battery (as we have seen on the example of the thermal qubit).

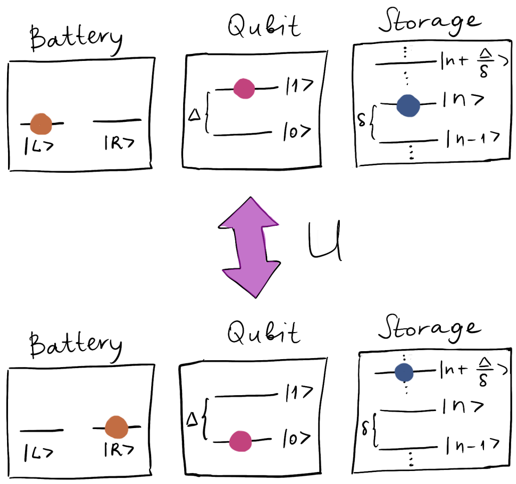

Figure 4: **The key transformation.** Essentially, the unitary provides an exchange of states shown above, which corresponds to lowering/lifting of energy of the energy battery.

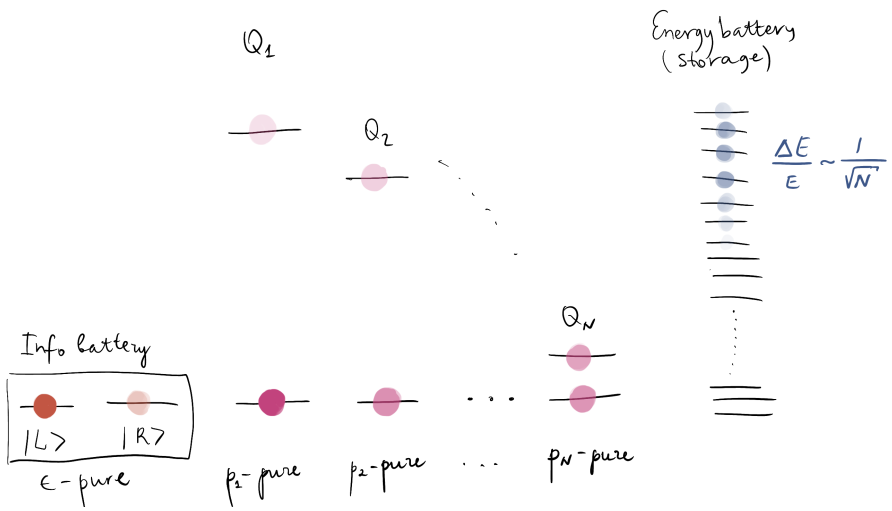

Figure 5: **The setting for N thermal qubits.** Here we order thermal qubits in the order of their energy gaps and then carry out the transformation on the Figure 4 for each qubit individually.

Tasks

1. Implement the work extraction protocol for 1 degenerate qubit information battery and 1 thermal qubit in an energy-conserving way, and store work in the energy battery (storage). Optimize the energy gap of the thermal qubit; compute the efficiency of the process and fluctuations in the battery.

2. Generalize the protocol for N thermal qubits. What is the final average energy in S, and the fluctuations?

3. Suppose now that you have one information qubit in a fully mixed state. Carry out an erasure operation on it, i.e. reset it to $|0\rangle$.

4. The engineers decided to replace the information battery, and now it consists of m degenerate qubits. Correct the work extraction protocol and implement it.

5. After all your achievements so far, you have been promoted to work on more ambitious projects. You are moved to another lab, and here the information battery can only be constructed out of qubits in an arbitrary state ρ_B. Work with what you are provided, and adjust your protocol: first for one such qubit, and then for m of them.
Hint: use a unitary scheme for information compression, for example, Schumacher compression [9]; a quick overview can be found in [10] (p 547).

References

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